Exam Statistical Mechanics
19 December 2023, 2pm

Dieterici equation of state [4 pts]

Besides the van der Waals model, a number of alternative equations to model gas-liquid systems were developed. One of these is the Dieterici equation of state

$$P = \frac{Nk_BT}{V - Nb}\, e^{-\frac{aN}{k_BTV}}$$

with $a, b > 0$ some parameters.

a) Show that at sufficiently low densities ($nb \ll 1$ and $na \ll k_BT$) the above equation reduces to the classical ideal gas law.

b) Calculate the low density behavior and show that it matches that of the van der Waals model to order n^2.

c) Show that the Dieterici model becomes unstable for $T < T_c$ and find T_c as a function of the parameters a and b.

d) Sketch the behavior of the isotherms P vs. $v = V/N$ for $T > T_c$ and $T < T_c$.

Quantum partition functions [4 pts]

A quantum system is characterized by the following energy spectrum $E_n = E_0 + n\gamma$ where $n = 0, 1, 2\ldots$ is a quantum number. In this system the n-th leven is $n+1$-degenerate, hence there is a non-degenerate ground state with energy E_0, two degenerate states with energy $E_1 = E_0 + \gamma$, three degenerate states with energy $E_2 = E_0 + 2\gamma$ and so on.

a) Calculate the partition function for this system and the average total energy $\langle E \rangle$. Plot $\langle E \rangle$ as a function of the temperature T and derive the low and high temperature limits.

b) Give an estimate of a characteristic temperature separating the classical from the quantum regime.

Identical particles wave function [4 pts]

We consider a system of non-interacting bosons characterized by a single particle spectrum with just two energy levels $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon > 0$. The two states are described by wavefunctions $\phi_1(\vec{q})$ and $\phi_2(\vec{q})$.

a) Find the wave function $\Psi(\vec{q}_1, \vec{q}_2, \vec{q}_3)$ associated to two bosons in the energy level ε_1 and one boson in ε_2. Normalize this correctly.

b) Find the grand canonical partition function $\Xi(\mu, T)$ for this system.

c) Find the value of $\mu(T)$ for which the average occupation number for the energy level ε_1 is twice as large as that of the energy level ε_2. Show that such μ exists only if the temperature is sufficiently large and find this value.

Blackbody spectrum [3 pts]

The usual Planck's law for blackbody radiation has the following form:

$$\varepsilon(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

This is the energy of frequency ω emitted from a black body at a temperature T.

The number of modes with frequency in the range $[\omega, \omega + d\omega]$ for a body of volume V is given by $g(\omega)d\omega$ with:

$$g(\omega) = \frac{V}{\pi^2 c^3}\omega^2$$

In d-dimensions the density of states generalizes to

$$g(\omega) = AV\omega^{d-1}$$

with A a proportionality constant.

How does the total emitted energy scale with the temperature in d dimensions?

[**Bonus points (+2)**] Calculate explicitely $g(\omega)$ in the two dimensional case.

Fermions [5 pts]

Calculate the variance in the occupation number of a state with energy ε_γ in a system of non-interacting fermions. Show that this is maximal for $\mu = \varepsilon_\gamma$ and calculate this maximal value.