Let X be a Banach space and $Y \subset X$ a closed vector subspace. We call Y a *complemented subspace* if and only if there exists a closed vector subspace $Z \subset X$ satisfying $Y \cap Z = \{0\}$ and $X = Y + Z$.

Part A. Prove that every closed subspace of a Hilbert space is a complemented subspace.

Part B. Let X be a Banach space and $Y \subset X$ a closed complemented subspace. Take a closed vector subspace $Z \subset X$ satisfying $Y \cap Z = \{0\}$ and $X = Y + Z$.

- Let $Y \oplus Z$ be the Banach space with $\|(y, z)\| = \|y\| + \|z\|$. Prove that the linear map $\theta : Y \oplus Z \to X : \theta(y, z) = y + z$ has a bounded inverse.
- Deduce the existence of $\alpha > 0$ such that $\|y + z\| \geq \alpha \|y\|$ for all $y \in Y$, $z \in Z$.

Part C. Let X be a Banach space and $Y \subset X$ a vector subspace. Prove that the following two statements are equivalent.

- Y is closed and complemented.
- There exists a bounded linear map $E : X \to X$ satisfying $E \circ E = E$ and $Y = E(X)$.