Exam Functional Analysis

January 17, 2013

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for **4 hours.** You are allowed to eat or drink.
- After **2 hours,** you hand in your solutions for questions 1 and 2. During the third and fourth hour, you work on questions 3 and 4, and you will have your oral exam about questions 1 and 2. **After 4 hours, the exam ends.**
- The exam is **open book.** This means that you may use
  - the lecture notes,
  - your own notes,
  - the two reference books.

  You are **not allowed to use**
  - any electronic equipment,
  - other books than the two reference books.
- This part of the exam counts for 12 of the 20 points. Every of the four questions has the same weight. The other 8 of the 20 points are attributed on the take home exam.

Write your name on every sheet that you hand in !

Good luck ! Stefaan Vaes

1. Let X be a Banach space and denote by $Y := \mathcal{F}(\mathbb{N}, X)$ the vector space of all functions from $\mathbb{N}$ to X.

   a) Define a seminorm topology on Y such that a net of functions $(f_i)_{i\in I}$ in Y converges to f in this seminorm topology if and only if f_i converges pointwize to f, meaning that

   $$\lim_{i\in I} \|f_i(k) - f(k)\| = 0 \quad \text{for every fixed } \; k \in \mathbb{N} \; .$$

   b) Let $\omega : Y \to \mathbb{C}$ be a linear map. Prove that the following two statements are equivalent.

   [i] The map ω is continuous.

   [ii] There exists an $n \in \mathbb{N}$ and $\omega_0, \ldots, \omega_n \in X^*$ such that

   $$\omega(f) = \sum_{k=0}^{n} \omega_k(f(k)) \quad \text{for all } \; f \in Y \; .$$

2. Let X be a seminormed space with its seminorm topology. Let $Y \subset X$ be a vector subspace and $x_0 \in X$. Prove that the following two statements are equivalent.

   a) x_0 belongs to the closure of Y.

   b) Every continuous linear map $\omega : X \to \mathbb{C}$ with $Y \subset \operatorname{Ker} \omega$ satisfies $\omega(x_0) = 0$.

   Hint. Use the Hahn-Banach separation theorem.

3. In the proof of Theorem 7.9, we find a subnet $(\mu_j)_{j\in J}$ of the sequence $(\omega_n)_{n\in\mathbb{N}}$ such that $(\mu_j)_{j\in J}$ converges in the weak* topology to $\mu \in \ell^\infty(\mathbb{Z})^*$.

   a) Is the sequence $(\omega_n)_{n\in\mathbb{N}}$ itself weak* convergent ? Prove your answer.

   b) Prove statements 1, 2 and 3 at the end of the proof of Theorem 7.9, page 75.

4. Let X be a Banach space and $T : X \to X$ a linear map satisfying $\|T(x)\| \leq \|x\|$ for all $x \in X$. Assume that $x_0 \in X$ is a nonzero vector satisfying $T(x_0) = x_0$. Prove that there exists $\omega \in X^*$ such that $\omega(x_0) = 1$ and $\omega(T(x)) = \omega(x)$ for all $x \in X$.